showed no phosphorescence of any kind even on the metal plate (compare paragraphs 4 and 5, Part I).

48. These observations at $-100^\circ$ confirm the view (paragraph 38) that the effect of low temperature is to increase the insulating power of the molecules, and thereby to enable some substances to store chemical energy which are unable to do so at all at higher temperatures, *e.g.*, barium platino cyanide; and to increase the storage capacity of other substances in which the secondary phosphorescence is very short-lived at higher temperatures, *e.g.*, potassium chloride and bromide. In the case of calcspar, the insulating capacity at ordinary temperatures is already so excellent, as shown by the persistence of secondary phosphorescence, that there is comparatively little gain at the lower temperature.

49. If, as seems most probable, the coloration of solids by the $\beta$ rays is due to the presence of ions, it is interesting to note that the different salts of potassium give perfectly distinct colours, the chloride being a red violet and the bromide and iodide a greenish blue. This difference is most likely due to the modification of the colour of the potassium ions by the presence of the haloid ions, chlorine, bromine, or iodine.]

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"Note on the Determination of the Volume Elasticity of Elastic Solids."* By C. Chree, Sc.D., LL.D., F.R.S. Received December 21, 1904,—Read February 2, 1905.

In a recent paper† Mr. A. Mallock gives an ingenious and simple method of determining the coefficient of volume elasticity (bulk modulus) of metals by direct observation of the extension of a hollow right circular cylinder under uniform internal pressure.

The method depends on a result of the mathematical theory which seems capable of being proved in a more direct and complete way, but which at the same time requires to be restricted by conditions to which Mr. Mallock does not seem to refer. Further, the method is only one of several which seem equally worthy of consideration.

When dealing with isotropic material I shall employ the following notation:—

$$E = \text{Young's modulus}, \qquad \eta = \text{Poisson's ratio},$$

$$k \equiv \tfrac{1}{3}E/(1-2\eta) = \text{bulk modulus}.$$

* [The main results of the first part of this paper (Case i) were worked out as a verification before Mr. Mallock's paper was printed; they were considered to have been sufficiently indicated in a footnote appended to the paper.—J. L.]

† 'Roy. Soc. Proc.,' vol. 74, p. 50.

Also $l$ will denote the length of the whole or a part of a generator of a right cylinder or prism, $\delta l$ its elastic extension.

When a hollow tube of isotropic material, bounded by co-axial right circular cylinders, is exposed to uniform pressures, viz., $p_i$ over the inner surface, radius $r_1$, and $p_e$ over the outer surface, radius $r_2$, and to uniform tension P over the flat ends, the elastic displacement parallel to the length, taken as axis of $z$, is given by

$$w = z\left\{\frac{\mathrm{P}}{\mathrm{E}} - \frac{2\eta}{\mathrm{E}}\frac{r_1^2 p_i - r_2^2 p_e}{r_2^2 - r_1^2}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).*$$

Case (i).—

$$p_e = 0, \quad \mathrm{P} = p_i r_1^2/(r_2^2 - r_1^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

$$\frac{\delta l}{l} = \frac{w}{z} = p_i \frac{r_1^2}{r_2^2 - r_1^2}\,\frac{1-2\eta}{\mathrm{E}} = \frac{p_i r_1^2}{r_2^2 - r_1^2}\,\frac{1}{3k} \quad \ldots\ldots\ldots \quad (3).$$

Putting for shortness

$$\delta l/l = \epsilon, \qquad r_2 - r_1 = t,$$

we have

$$k = \frac{p_i r_1}{3t\epsilon}\,\frac{r_1}{r_2 + r_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

When the thickness, $t$, of the cylinder wall is so small that $t/r_1$ is negligible compared to unity,

$$k = p_i r_1/6t\epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

Case (ii).—

$$p_i = 0, \quad \mathrm{P} = -p_e r_2^2/(r_2^2 - r_1^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

$$\frac{\delta l}{l} = \frac{w}{z} = -p_e \frac{r_2^2}{r_2^2 - r_1^2}\,\frac{1}{3k} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

Thus, with the same notation as in Case (i),

$$k = -\frac{p_e r_2}{3t\epsilon}\,\frac{r_2}{r_2 + r_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8).$$

When $t/r_2$ is negligible compared to unity,

$$k = -p_e r_2/6t\epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9).$$

Case (iii).—

$$p_i = p_e = -\mathrm{P} = p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10).$$

$$\delta l/l = w/z = -p/3k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11),\dagger$$

and so

$$k = -p/3\epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12).$$

Case (i) is really that considered by Mr. Mallock. Taking a cylindrical tube with terminal caps, he applies a uniform internal pressure $p_i$. This gives on each cap an outwardly directed pressure, whose total amount is $\pi r_1^2 p_i$. This leads to a longitudinal tension in the cylinder

* 'Amer. Math. Soc. Bull.,' 2nd series, vol. 7, 1900, p. 141.

† *Cf.* 'Phil. Mag.,' December, 1901, Eqn. (67), p. 594.

wall. Denoting the mean value of this tension per unit area by P, we have, since the cross-section of the material is $\pi(r_2^2 - r_1^2)$,

$$P\pi(r_2^2 - r_1^2) = p_i \pi r_1^2, \quad \text{or} \quad P = p_i r_1^2/(r_2^2 - r_1^2).$$

The first limitation above referred to is this: The longitudinal tension exerted on the cylindrical wall by the terminal caps will not in reality be uniformly distributed over the terminal areas $\pi(r_2^2 - r_1^2)$; thus the solution is reliable only when the principle of equipollent systems of loading* is applicable. This means that portions of the tube near the ends should be excluded from the solution, the portions being shorter the thinner the cylindrical wall.

The second limitation is that Mr. Mallock starts by assuming the tube wall very thin, and arrives at a formula which is presumably (5), and which at all events possesses the same limitations.† The formula is applied, however, by Mr. Mallock on his p. 52 to cases in which $t/r_1$ varies from about 1/19 to 1/5. The application of (5) under the circumstances would mean an error of from $2\frac{1}{2}$ to 10 per cent. in the value of $k$.

Case (ii) is parallel in every respect to Case (i) and would apply,

* Todhunter and Pearson's 'History of Elasticity,' vol. 2, Arts. [8], etc.

† The primary formula, to which Mr. Mallock's calculations seem to lead him, as given on his p. 51, is

$$\kappa_x = Pr/2te_x,$$

from which he apparently derives $\kappa = Pr_2/6te$.

P represents the internal pressure. I do not follow Mr. Mallock's proof of the formulæ. But his $\kappa$ and $e$ must, I think, be respectively equivalent to $k$ and $\epsilon$ of the present paper.

An attempt to ascertain the true significance of Mr. Mallock's $r_2$ from the numerical results on his p. 52, seemed to show that some errors of calculation must exist, independently of the precise meaning of $r_2$. It thus appeared desirable to recalculate the results from the exact formula (4). They appear in the following table, alongside of the values given by Mr. Mallock. The pressure applied was in all cases "400 lbs. per square inch."

| Material. | $l$. | Outside diameter. | $t$. | State of material. | $\delta l \times 10^3$. | $10^{-11} \times k$ in C.G.S. units. | |
|---|---|---|---|---|---|---|---|
| | | | | | | Mallock. | By (4). |
| | in. | in. | in. | | in. | | |
| Steel .... | 60 | 0·75 | 0·0190 | Hard | 2·8 | 18·4 | 18·0 |
| | | | | Annealed | 2·75 | 18·2 | 18·3 |
| Brass.... | 50 | 0·415 | 0·0185 | Hard | 2·12 | 11·05 | 10·6 |
| | | | | Annealed | 2·09 | 10·75 | 10·7 |
| Copper .. | 50 | 0·4485 | 0·0382 | Hard | 0·562 | 23 | 18·1 |
| | | | | Annealed | 0·71 | 16·2 | 14·3 |

There is an obvious omission of decimals in Mr. Mallock's values for brass, which has been corrected above.

under similar limitations, to a cylindrical tube closed by caps and exposed to uniform external pressure $p_e$. The tube exposed to pressure would have to be contained in a strong vessel, having a glass roof or side. This might present disadvantages, but the test tube itself would be less exposed to temperature changes than in Case (i) and might be of a very simple type; so that once the containing vessel was built experiments on a variety of materials would be simple.

Case (iii) is only a special instance of the obvious result that when an isotropic solid of any shape, bounded by any number of surfaces, is exposed to pressure $p$, the same at every point of each and all of its bounding surfaces, *every* line in it whose original length is $l$ suffers a shortening given by $\delta l/l = -p/3k$.

In Case (iii) any body will serve the purpose, but a long rod would be the natural form to adopt. This means a simpler test object than in Cases (i) and (ii), and there is the further recommendation that the mathematical solution is exact right up to the extremities of the object. The compensating disadvantage is that it requires a very long test object, or a very high pressure, to give sufficient extension, unless extremely refined methods of measurement exist. In Cases (i) and (ii) by using a thin walled tube one has much greater sensitiveness.

By subjecting a cylindrical tube successively to internal and external pressures, and determining values for $k$ by both the methods (i) and (ii), interesting information could be obtained as to whether the bulk modulus is or is not the same for extension and compression. Removing the caps, and determining E by simple longitudinal tension, one would thus determine fully the elastic properties of the material, assuming it isotropic.

In any case $\delta l$ should be measured from the equilibrium position of the test object after it is supported, and if it is hollow and exposed to internal fluid pressure after it has been filled with liquid.

Results in some respects more general than the preceding are obtained very simply from the formulæ which I have given for the mean change of length in elastic solids under any given system of loading.

Not confining ourselves to isotropy, suppose first that the material is merely symmetrical with respect to three rectangular planes, which we may suppose perpendicular to the axes of $x$, $y$ and $z$.

Let $E_3$ be Young's modulus for traction parallel to the axis of $z$, and $\eta_{31}$, that one of the (six) Poisson's ratios which answers to traction parallel to $z$ and contraction (strain) parallel to $x$.

Let $g \equiv d\gamma/dz$ represent the extensional strain parallel to $z$. Then in a body of any shape exposed to surface forces whose components at any point parallel to $x$, $y$, and $z$ are denoted by F, G, H, the mean value $\bar{g}$ of $g$ throughout the whole volume $v$ is given by

$$E_3 v\bar{g} = \iint (Hz - \eta_{31} Fx - \eta_{32} Gy)\, dS \quad \ldots\ldots\ldots\ldots \quad (13).*$$

Confining ourselves to the case where the material is symmetrical, about $0z$, we have

$$\left.\begin{matrix} \eta_{31} = \eta_{32} = \eta, \ \text{say}, & \eta_{12} = \eta_{21} = \eta', \\ \eta_{13} = \eta_{23} = \eta'', \ \text{say}, & E_2 = E_1 = E' \end{matrix}\right\} \ldots\ldots\ldots \quad (14).$$

Also, writing E for $E_3$ it may be shown that

$$\eta''/E' = \eta/E \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15).$$

Suppose, now, the solid to be a hollow right prism exposed to uniform tension P over its flat ends, S, and to uniform pressures, $p_i$ over its inner surface $S_i$, and $p_e$ over its outer surface $S_e$, then

$$Ev\bar{g} = \iint PzdS + \eta \iint p_i (\lambda_i x + \mu_i y)\, dS_i + \eta \iint p_e (\lambda_e x + \mu_e y)\, dS_e \ldots \quad (16),$$

where $\lambda$, $\mu$, $\nu$ are the direction cosines of the normal drawn *from* the solid, and the suffix denotes the surface.

Taking the origin in one end of the prism, and supposing its length $l$, we have $z = 0$ over the one flat end and $z = l$ over the other; thus S denoting the area of one end,

$$\iint PzdS = PlS = Pv,$$

where $v$ is the volume of the *material.*

Over the two prismatic surfaces

$$\lambda_i x + \mu_i y = -\varpi_i, \quad \text{and} \quad \lambda_e x + \mu_e y = \varpi_e,$$

where $\varpi_i$ and $\varpi_e$ represent the perpendiculars from some internal fixed point on the tangent planes to the two surfaces. But obviously

$$\iint \varpi_i dS_i = 2v_i, \qquad \iint \varpi_e dS_e = 2v_e,$$

where $v_i$ and $v_e$ are the volumes included within $S_i$ and $S_e$ respectively. Thus (16) gives at once

$$Ev\bar{g} = Pv + 2\eta\,(p_e v_e - p_i v_i).$$

Clearly $\bar{g} = \delta\bar{l}/l$, where $\delta\bar{l}$ represents the mean change in length of all the longitudinal "fibres" of which the prism is composed. Thus

$$\delta\bar{l}/l = \frac{P}{E} + \frac{2\eta}{E}\left(\frac{p_e v_e - p_i v_i}{v}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17).$$

As in the isotropic circular cylinder, there are three principal cases:

$$\text{(i)} \quad p_e = 0, \quad Pv = p_i v_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (18),$$

$$\delta\bar{l}/l = p_i (v_i/v)/3k_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (19);$$

* 'Camb. Phil. Soc. Trans.,' vol. 15, Eqn. (15), p. 317.

$$\text{(ii)}\quad p_i = 0, \quad \mathrm{P}v = -p_e v_e \quad \ldots\ldots\ldots\ldots \quad (20),$$

$$\delta\bar{l}/l = -p_e(v_e/v)/3k_3 \quad \ldots\ldots\ldots\ldots \quad (21);$$

$$\text{(iii)}\quad p_i = p_e = -\mathrm{P} = p \quad \ldots\ldots\ldots\ldots \quad (22),$$

$$\delta\bar{l}/l = -p/3k_3 \quad \ldots\ldots\ldots\ldots \quad (23).$$

Here

$$k_3 \equiv \tfrac{1}{3}\mathrm{E}/(1-2\eta) \quad \ldots\ldots\ldots\ldots \quad (24).$$

Noticing that $\dfrac{v_i}{v} = \dfrac{\text{Cross-section within } \mathrm{S}_i}{\text{Section of prism wall}}$,

and similarly for $v_e$, it is obvious that Cases (i) and (ii) answer respectively to uniform internal and external pressures in a hollow prism whose ends are covered by caps.

The above results are true—under the same limitations as for the isotropic cylinder—whatever be the shapes of the prismatic sections, and irrespective of whether they be similar to one another or not. If, however, the prismatic walls be thick, or the contours of the cross sections be irregular and dissimilar, there *may* be, so far as the above proof is concerned, considerable differences between the longitudinal alteration of different "fibres," and to obtain a satisfactory observational value for $\delta\bar{l}$ might be troublesome.

In the case of a right circular cylinder whose two surfaces are co-axial there cannot, of course, be any buckling, and if the walls are thin, $\delta\bar{l}$ will be given satisfactorily by measurement of a single generator.

The elastic modulus $k_3$ defined by (24) represents (*mean* surface pressure ÷ reduction in volume), in a unit cube when only the faces perpendicular to $z$ are under pressure, the remaining four faces being unstressed. It is distinct from the true bulk modulus $k$, answering to uniform pressure the same in all directions, which is given by

$$\frac{1}{k} = \frac{1-2\eta}{\mathrm{E}} + 2\left(\frac{1}{\mathrm{E}'} - \frac{\eta}{\mathrm{E}} - \frac{\eta'}{\mathrm{E}'}\right) \quad \ldots\ldots\ldots\ldots \quad (25),$$

or

$$\frac{1}{k} = \frac{1}{k_3} + 2\left(\frac{1-\eta'}{\mathrm{E}'} - \frac{1-\eta}{\mathrm{E}}\right) \quad \ldots\ldots\ldots\ldots \quad (26).$$